MARKOV PARTITIONS FOR ANOSOV FLOWS ON n-DIMENSIONAL MANIFOLDS

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ABSTRACT

In this work we construct a Markov partition for transitive Anosov flows, such that the measure of the boundary of the partition is zero. Symbolic dynamics for these flows is also developed.

In this paper we construct a Markov partition for transitive Anosov flows (which we call C-flows) on smooth Riemann manifolds. A partition of this type was constructed by Adler and Weiss in [1] for a hyperbolic automorphism of a two-dimensional torus. Then Sinai [13], [14] used successive approximations to define and construct a Markov partition for arbitrary C-diffeomorphisms. Bowen [5] modified Sinai's definition and, using the same method, constructed a Markov partition for the Axiom A diffeomorphisms of Smale. In the case of C-flows on three-dimensional manifolds, a construction of Markov partitions was given in [10]. Bowen then [7] carried over his proof for diffeomorphisms to the case of A-flows. In Section 2 we shall briefly describe the successive stages of Bowen's construction, omitting the proofs of certain intuitively obvious facts which are presented exhaustively in [5]. In Section 3 we shall prove that for C-flows of class C^2 the boundary of the elements of the Markov partition has Lebesgue measure 0. This was proved for C-diffeomorphisms by Sinai [41]. No proof of this fact is given in Bowen's paper.

In Section 4 we study symbolic dynamics for a C-flow $\{T^t\}$ on W. The Markov partition enables us to construct a special flow S^t (see [12]) in $\tilde{W} = \{(\omega, t).$ $\omega \in \Omega,\ 0 \leq t < F(\omega)\}$, over the space Ω of sequences of an intrinsic Markov chain (see [9]) with the natural topology, a shift σ in Ω and a positive function

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$F(\omega)$ satisfying a Hölder condition, in such a way that there is a continuous mapping $\phi: \tilde{W} \to W$ such that $T^t\phi = \phi S^t$, and σ is a topological mixing (see also [6]). This special representation enables us to consider for T^t the invariant Gibbs measures constructed by Sinai in [15]. With these measures, T^t is a K-flow. Among such measures, in particular, we find the so-called smooth invariant measures which induce on the elements of a measurable contracting or expanding partition a conditional measure equivalent to Riemannian volume. These measures were constructed for C-diffeomorphisms in [13] and for three-dimensional C-flows in [11]. In the case of a geodesic flow on a manifold of negative curvature, these measures coincide with invariant Riemannian volume.

1. Notation and definitions

Let W^n be an n-dimensional compact Riemannian manifold of class C^∞, $\{T^t\}$ a C-flow on W^n, $\Gamma_c(\Gamma_e)$ its k-dimensional contracting (l-dimensional expanding) foliation, $k + l + 1 = n$, $G_c(G_e)$ the $(k+1)$-dimensional ($(l+1)$-dimensional) foliation into leaves: $G_c = \bigcup_{t=-\infty}^{\infty} T^t\Gamma_c$ ($G_e = \bigcup_{t=-\infty}^{\infty} T^t\Gamma_e$). It is assumed that all leaves of the foliations Γ_c and Γ_e are dense in W^n.

$\Gamma_c^\delta(x)$ will denote the closed δ-ball in $\Gamma_c(x)$ centered at x. According to [2],

$$d_c(T^t y, T^t z) < c\lambda^t d_c(y,z) \text{ for } y,z \in \Gamma_c(x)$$

$$d_e(T^{-t} y, T^{-t} z) < c\lambda^t d_e(y,z) \text{ for } y,z \in \Gamma_e(x)$$

where $c > 0$, d_c, d_e denote the Riemann metrics on Γ_c, Γ_e, $0 < \lambda < 1$.

It is shown in [16] that there exist $\varepsilon_0 > 0$, $\gamma > 0$ such that for all x, $w \in \Gamma_c^\gamma(x)$, $v \in G_e^\gamma(x)$, the ε_0-balls $\Gamma_c^{\varepsilon_0}(v)$ and $G_e^{\varepsilon_0}(w)$ intersect in exactly one point $[v,w]$ and the mapping $(v,w) \to [v,w]$ of the direct product $\Gamma_c^\gamma(x) \times G_e^\gamma(x)$ onto a neighborhood of the point x in W^n is a homeomorphism. The mapping $\pi_c: z \to [z,y]$, $z \in G_e(x)$, $y \in \Gamma_c^\gamma(x)$, is a homeomorphism of $G_e^\gamma(x)$ onto a neighborhood of y in the leaf $G_e(y)$. We call the mapping π_c the c-isomorphism. The set $A' = [A,y]$ of all points $[z,y]$, $z \in A$, is said to be c-isomorphic to A. Similarly, the mapping $\pi_e: y \to [z,y]$ is called the e-isomorphism.

The notation $A \subset \Gamma_c, \Gamma_e, G_c, G_e$ will mean that A is a subset of a leaf of the foliation in question.

A subset A of a leaf is said to be admissible if in the metric of the leaf (1) $\mathrm{Int} A \neq \emptyset$. $\overline{\mathrm{Int} A} = A$; (2) the measure of the boundary ∂A is zero.

Let A_1 and A_2 be admissible subsets of a leaf $G_e(x)$.

DEFINITION 1.1. A_1 and A_2 are said to be trajectory isomorphic if one can define on A_1 a continuous function $q(x)$, $x \in A_1$, satisfying the following conditions:

(1) $q(A_1 \cap A_2) = 0$;

(2) $\bigcup_{x \in A_1} T^{q(x)} x = A_2$;

(3) the mapping $\psi: x \to T^{q(x)}x$ is a homeomorphism.

Let $A \subset \Gamma_e(x_0)$ be an admissible subset of the leaf $\Gamma_e(x_0)$ and B its c-isomorphic image on $G_e(x_1)$. For C-flows, the foliations Γ_c and Γ_e generally form a non-integrable pair (see [2]), and therefore the set B is not a subset of a leaf of Γ_e. However, it can be shown that for every point $z \in B$ there is a subset B' of $\Gamma_e(z)$ which contains z and is trajectory isomorphic to B. If the C-flow is also of class C^2, then the foliations Γ_c, Γ_e, G_c, G_e satisfy a Hölder condition of positive order (see [3], [8]) and have the absolute continuity property ([2], [4]), so that in this case the subset B' is admissible in $\Gamma_e(z)$ and the function $q(y)$, $y \in B'$, effecting the trajectory isomorphism between B and B', sat sfies a Hölder condition of positive order (see Section 3). In accordance with Definition 1.1 we introduce the notation $B' = \psi_z^{-1}B$.

We now consider admissible subsets $C \subset \Gamma_e^\gamma(x)$ and $D \subset \Gamma_c^\gamma(x)$.

DEFINITION 1.2. The set $P = [C,D]$ is called a c-parallelogram in W^n if $[y,z] \in P$ for all $y \in C$, $z \in D$. The diameters of the sets C and D are called the sizes of P.

Any c-parallelogram is the union of c-isomorphic sets $\{[C,z] = C(z),\ z \in D\}$ on the one hand, and of e-isomorphic sets $\{[y,D] = D(y),\ y \in C\}$, on the other; these sets are subsets of leaves of Γ_c. We denote $\partial_c P = \bigcup_{y \in \partial C} [y,D]$, $\partial_e P = \bigcup_{z \in \partial D}[C,z]$, $\partial P = \partial_c P \cup \partial_e P$. For $x \in [C,z]$, $x \in [y,D]$, we denote $[C,z] = C(x)$, $[y,D] = D(x)$. It will be convenient to write $P = [C,D]$, where $C = C(x)$, $D = D(x)$ for some $x \in P$.

Let $P = [C_P, D_P]$ and $Q = [C_Q, D_Q]$ be two c-parallelograms, $P \cap Q \neq \varnothing$ and $x \in P \cap Q$. Let $D_{PQ}(x) = D_P(x) \cap D_Q(x) \subset \Gamma_c(x)$ and $C_{PQ}(x) = \psi_x^{-1} C_P(x) \cap \psi_x^{-1} C_Q(x) \subset \Gamma_e(x)$. The notation $P \cap Q \neq \varnothing$ will always mean that $\operatorname{Int} D_{PQ}(x) \neq \varnothing$ in $\Gamma_c(x)$ and $\operatorname{Int} C_{PQ}(x) \neq \varnothing$ in $\Gamma_e(x)$. Let $C_P^Q(x)$ denote the ψ_x-image of $C_{PQ}(x)$ on $C_p(x)$ and $C_Q^P(x)$ the ψ_x-image of $C_{PQ}(x)$ on $C_Q(x)$. The parallelogram $P_Q = [C_P^Q(x), D_{PQ}(x)] \subset P$ ($Q_P = [C_Q^P(x), D_{PQ}(x)] \subset Q$) will be called the projection of Q on P (of P on Q). Two parallelograms P_1 and P_2 are said to be parallel if there exists a continuous function $\tau(x)$ on P_1 such that $\bigcup_{x\in P_1} T^{\tau(x)}x = P_2$. The projections P_Q and Q_P are obviously parallel.

Given a parallelogram $P = [C_P, D_P] \supset C_P$, we define a continuous positive function $\tau(x)$ on C_P.

DEFINITION 1.3. Any set V of the form $V = \bigcup_{x\in C_n} \bigcup_{t=0}^{\tau(x)} T^t D_P(x)$ is called a c-parallelepiped in W^n; P will be called the lower face of V.

DEFINITION 1.4. A finite system of c-parallelepipeds $\{V_i,\ i = 1,\cdots,k\}$ in W^n will be called a partition if $W^n = \bigcup_{i=1}^{k} V_i$ and $V_i \cap V_j = \partial V_i \cap \partial V_j$, $i \neq j$.

Henceforth we shall speak simply of parallelograms instead of c-parallelograms; the entire discussion can be carried out in symmetric fashion for e-parallelograms.

Let $\mathfrak{A}$ be a system of parallelograms. We shall say that $\mathfrak{A}$ is complete if for every point $w \in W^n$ there exists an interval on the trajectory of w whose endpoints lie in elements of $\mathfrak{A}$. Let $\mathfrak{A} = \{P_i,\ i = 1,\cdots,k\}$, $P_i = [C_i, D_i]$, $P_i \cap P_j = \varnothing$, $i \neq j$, be a finite complete system of parallelograms, and let $M = M_{\mathfrak{A}}$ be the set-theoretic union of the parallelograms $\{P_i\}$ with the induced topology. Let $l(x)$, $x \in M$, denote the length of the interval on the trajectory of the flow $\{T^t\}$ extending from x in the positive direction to its first intersection x' with M. It is obvious that $0 < l_0 \leqq l(x) < \infty$. We denote by $f_m = f_{\mathfrak{A}}$ the one-to-one mapping of M onto itself defined by $x \to x'$. Following Bowen, we define

DEFINITION 1.5. A system $\mathfrak{A}$ is said to be Markovian for the C-flow $\{T^t\}$ if, whenever $x \in \operatorname{Int} P_i \cap f^{-1}(\operatorname{Int} P_j)$,

$$f(\operatorname{Int} D_i(x)) \subset D_j(f(x)) \tag{1.2}$$

$$f(C_i(x)) \supset \operatorname{Int} C_i(f(x)). \tag{1.3}$$

DEFINITION 1.6. A partition μ into c-parallelepipeds $\{V_i\}$ is called a Markov partition for $\{T^t\}$ if the system $\mathfrak{B}(\mu)$ of lower faces of the parallelepipeds $\{V_i\}$ is Markovian.

It follows from Definition 1.3 that the function $l_{\mathfrak{B}}$ is continuous on the open parallelograms $P = [C_P, D_P] \in \mathfrak{B}$ and constant on $[z, D_P]$, $z \in C_P$. Consider $\partial_c\mathfrak{B} = \bigcup_{P\in\mathfrak{B}} \partial_c P$, $\partial_e\mathfrak{B} = \bigcup_{P\in\mathfrak{B}} \partial_e P$, $\partial\mathfrak{B} = \partial_c\mathfrak{B} \cup \partial_e\mathfrak{B}$.

PROPOSITION 1.1. *If $x \in \partial_c\mathfrak{B}$, $y \in \partial_e\mathfrak{B}$, then $f(x) \in \partial\mathfrak{B}$ and $f^{-1}y \in \partial\mathfrak{B}$.*

PROOF. Let $y \in \partial_e P$ and $z = f^{-1}y \in \operatorname{Int} Q$, $P, Q \in \mathfrak{B}$. Let $O_c(z)$ be a neighborhood of z in $\operatorname{Int} D_Q(z)$. Since $l_{\mathfrak{B}}(z)$ is constant on $\operatorname{Int} D_Q(z)$, it follow that

$f(O_c(z))$ is a neighborhood of y in $\Gamma_e(y)$. Let $u \in O_c(z)$ such that $f(u) \in \operatorname{Int} P$. By the Markov property, $f(\operatorname{Int} D_Q(u)) \subset D_P(f(u))$, and consequently $f(O_c(z)) \subset D_P(f(u))$, i.e., $y \in \operatorname{Int} D_P(f(u))$, which contradicts the assumption $y \in \partial_e P$. Therefore $z \in \partial Q$. The proof that $f(x) \in \partial\mathfrak{B}$ is analogous.

Now let $M = M_{\mathfrak{B}}$ and $\tilde{M} = M \setminus \bigcup_{t=-\infty}^{\infty} T^t \partial\mathfrak{B}$.

PROPOSITION 1.2. *Let $w_0 \in M$ be a periodic point of the C-flow $\{T^t\}$. Then either $w_0 \in \partial\mathfrak{B}$ or $w_0 \in \tilde{M}$.*

PROOF. Since w_0 is periodic, there exists $k > 0$ such that $f^k(w_0) = w_0$. Suppose that $w_0 \notin \tilde{M}$. Then there exists m, $0 \leqq m < k$, such that $f^m w_0 \in \partial\mathfrak{B}$.

If $f^m w_0 \in \partial_c\mathfrak{B}$, it follows from the foregoing that $w_0 = f^{k-m}(f^m w_0) \in \partial\mathfrak{B}$ $(k - m > 0)$, while if $f^m w_0 \in \partial_e\mathfrak{B}$ then $w_0 = f^{-m}(f^m w_0) \in \partial\mathfrak{B}$. Consequently, if $w_0 \notin \tilde{M}$ then $w_0 \in \partial\mathfrak{B}$.

2. Construction of Markov partition

In this section we prove the following

THEOREM 2.1. *For every $\varepsilon > 0$, the C-flow $\{T^t\}$ has a Markov partition μ into c-parallelepipeds $\{V_i\}$, the sizes of whose lower faces are at most ε, such that the function $l_{\mathfrak{B}(\mu)}$ satisfies a Hölder condition of positive order on every continuity component.*

The validity of the Hölder condition will be proved in Section 3; the present section is devoted to construction of the Markov partition.

Let $\mathfrak{A}^0 = \{A_1^0, \cdots, A_k^0\}$ be a complete finite system of parallelograms, $A_i^0 = [C_i^0, D_i^0]$, $C_i^0 = \Gamma_e^\alpha(x_i)$, $D_i^0 = \Gamma_c^\alpha(x_i)$, $A_i^0 \cap A_j^0 = \varnothing$ for $i \neq j$, $0 < \alpha < \min(\varepsilon, \gamma)$. For $x \in D_i^0$, we consider $\psi_x^{-1} C_i^0(x)$ and the function $q_x(z)$, $z \in \psi_x^{-1} C_i^0(x)$, defining the trajectory isomorphism between $C_i^0(x)$ and $\psi_x^{-1} C_i^0(x)$. We set:

$$q = \max_{i=1,\cdots,k} \; \min_{x \in D_i^0} \; \max_{z \in \psi_x^{-1} C_i^0(x)} |q_x(z)|, \quad M = M_{\mathfrak{A}^0}$$

$$\min_{x \in M} l_{\mathfrak{A}}(x) = L < \infty, \min_{x \in M} l(x) = l > 0$$

$$L < \tau < L + l, \quad V_i^0 = \bigcup_{t=0}^{\tau} T^{-t} A_i^0$$

$$T^{-t} A_i^0 = A_{it}^0 = [C_{it}^0, D_{it}^0].$$

Suppose that the sets $\{\operatorname{Int} V_i^0, \; i = 1, \cdots, k\}$ cover W^n. Let α be so small that

$2q \leqq l$ and there exists $\alpha < \delta < \min(\varepsilon, \gamma, \varepsilon_0)$ such that the diameters of the sets $\{T^{-t}D_i^0(y),\ y \in \Gamma_e^\delta(x_i),\ t \in [-2q, \tau + 2q]\}$ and $\{T^{-t}\psi_z^{-1}C_i^0(z),\ z \in \Gamma_c^\delta(x_i),\ t \in [-2q, \tau + 2q]\}$ are at most $\delta/2$.

As mentioned in the introduction, we shall state here several lemmas whose full proofs are given in [5]. These lemmas illustrate the successive approximation of the leaves $\{C_i^0\}$ and $\{D_i^0\}$ to the Markov property. The procedure (Lemmas 2.1 and 2.2) consists in applying to D_k^0 a sufficiently strong expansion $T^{-m}D_k^0$ (where m is large), adding to the leaf $T^{-m}D_k^0$ sets $D_{it}^0(z) = [z, D_{it}^0]$, $t \in [0, \tau]$, $z \in C_{i_t}^0$, which cut the boundary $\partial T^{-m}D_k^0$ at a certain point z in such a way that $T^{-m}C_k^0(T^m z) \subset C_{it}^0(z)$ and $T^{-m}C_k^0(T^m z)$ remains at a certain fixed distance from the boundary of $C_{i_t}^0(z)$. However, the relation $T^{-m}C_k^0(T^m z) \subset C_{it}^0(z)$ is not rigorously correct, since in general, neither $T^{-m}C_k^0(T^m z)$ nor $C_{it}^0(z)$ is a subset of a leaf of Γ_e; we therefore project them by a trajectory isomorphism ψ onto $\Gamma_e(z)$ and write $\psi_z^{-1}T^{-m}C_k^0(T^m z) \subset \psi_z^{-1}C_{i_t}^0(z)$. The set obtained after adding the sets $D_{it}^0(z)$ is contracted by applying T^m and we obtain the first approximation D_k^1. The procedure is then repeated for $\{D_k^1\}$ and $D_{i_t}^1(z)$, and so on (Lemma 2.3). In the limit we obtain sets $\{D_i\}$ such that the Markov property holds for all $y \in \Gamma_e^\delta(x_i)$: $T^{-m}[y, D_i] \supset [T^{-m}y, D_{jt}(T^{-m}y)]$ for some j and $t \in [0, \tau]$ (Lemma 2.4). The sets D_i have the property $D_i = \overline{\mathrm{Int}\, D_i}$. In Section 3 we shall prove that for a C-flow of class C^2 the measure of the boundary ∂D_i is zero and consequently the D_i are admissible sets. An analogous construction is applied to C_i^0, the only difference being that the sets $[C_{it}^0, z]$ which we must add to $T^m C_i^0$ are not subsets of the leaves of Γ_e and therefore we add certain trajectory-isomorphic images on $\Gamma_e(T^m C_i^0)$ (Lemmas 2.2′, 2.3′, 2.4′). This introduces a correction q in consideration of the interval $[-q, \tau + q]$, since under a trajectory projection, every point is translated along its orbit by at most q. The limit set C_i does not leave the δ-neighborhood $\Gamma_e^\delta(x_i)$ and therefore the Markov property remains valid for D_i. Considering now the parallelograms $A_i = [C_i, D_i]$, we see that both Markov properties hold for $y \in A_i$:

$$T^{-m}[y, D_i] \supset [T^{-m}y, D_{jt_1}]$$

and

$$T^m\psi_y^{-1}[C_i, y] \supset \psi_{T^m y}^{-1}[C_{kt_2}, T^m y]$$

for some $t_1, t_2 \in [-q, \tau + q]$ (Lemma 2.5).

The parallelograms $\{T^{-t}A_i, t \in [0, \tau],\ i = 1, \cdots, k\}$ cover W^n. We then construct a new cover ν by parallelograms, decomposing each $T^{-t}A_i$ in such a way that, if two of the new parallelograms $P, Q \in \nu$ intersect, $\mathrm{Int}\,P \cap \mathrm{Int}\,Q \neq \varnothing$, then they are parallel (see Sec. 1) (relations (2.1), (2.1′), Lemmas 2.6 and 2.7).

Every element B of the new partition ν on $\{A_i\}$ is again decomposed, projecting onto B all parallelograms $Q \in \nu$ which intersect with the translates $T^{-t}B$ on the entire interval $t \in [0, m]$ (this decomposition is denoted by $\alpha(B)$). The Markov property of the parallelograms $\{A_i\}$ for $t = m$ guarantees the Markov property of this last partition (Lemma 2.8).

LEMMA 2.1. *There exist $a > 0$ and a mapping F:*

$$W^n \to J \times I, x \to (i,t),\ i \in \{1,2,\cdots,k\} = J;\ t \in [0,\tau] = I$$

such that $x \in A^0_{F(x)}$, $\Gamma^a_c(z) \subset D^0_{F(x)}(z)$ for all $z \in C^0_{F(x)}(x)$, and $\psi_y^{-1}C^0_{F(x)}(y) \supset \Gamma^a_e(y)$ for all $y \in D^0_{F(x)}(x)$.

Now choose m so large that $c\,\sum_{j=1}^{\infty}\lambda^{mj} < a/\gamma$ and set $g = T^{-m}$, $\beta = \lambda^m$.

LEMMA 2.2. *Let $i = 1,\cdots,k$. We can find points $\{y_{ij}, j = 1,\cdots,s_i\}$ in $\Gamma^\delta_c(x_i)$ such that for $z_{ij} = g(y_{ij})$:*

a) $g(D^0_i) \cap D^0_{F(z_{ij})}(z_{ij}) \neq \varnothing$,

b) $g(D^0_i) \subset \bigcup_{1 \leqq j \leqq s_i} D^0_{F(z_{ij})}(z_{ij})$.

Next set $D^n_i = \bigcup_{1 \leqq j \leqq s_i} g^{-1}D^{n-1}_{F(z_{ij})}(z_{ij})$.

LEMMA 2.3. *For $n \geqq 1$ and $y \in \Gamma^\delta_e(x_i)$, $[y, D^n_i]$ has dense interior in $\Gamma^\gamma_c(y)$ and*

$$[y, D^n_i] \subset \Gamma_c^{(1+\cdots+\beta^n)\delta/2}(y) \subset \Gamma^\delta_c(y).$$

Denote $D_i = \overline{\bigcup_{n \geqq 0} D^n_i} \subset \Gamma^\delta_c(x_i)$, $D_i = \overline{\mathrm{Int}\,D_i}$ in $\Gamma^\delta_c(x_i)$.

LEMMA 2.4. *If $z \in [\Gamma^\delta_e(x_i), D_i]$, then, for some $j(z) \in J$, $t(z) \in [-q, \tau + q]$ and $C^0_{j(z)t(z)}, D_{j(z)t(z)}]$ containing $g(z)$:*

$$g[z, D_i] \supset [g(z), D_{j(z)t(z)}]$$

and

$$j(z) = j([x_i, z]).$$

Similarly, working with g^{-1} and C^0_i we have:

LEMMA 2.2′. *There are points u_{ir}, $r = 1,\cdots,p_i$ in $\Gamma^\delta_e(x_i)$ such that for $w_{ir} = g^{-1}(w_{ir})$:*

a) $g^{-1}(C_i^0) \cap \psi_{w_{ir}}^{-1} C^0_{F(w_{ir})}(w_{i_r}) \neq \emptyset$,

b) $g^{-1}(C_i^0) \subset \bigcup_{1 \leqq r \leqq p_i} \psi_{w_{ir}}^{-1} C^0_{F(w_{ir})}(w_{i_r})$.

We set $C_i^1 = \bigcup_{1 \leqq r \leqq p_i} g\psi_{w_{ir}}^{-1} C^0_{F(w_{ir})}(w_{i_r})$,

$$C_i^n = \bigcup_{1 \leqq r \leqq p_i} g\psi_{w_{ir}}^{-1} C^{n-1}_{F(w_{ir})}(w_{ir}).$$

LEMMA 2.3′. *For $n \geqq 1$ and $z \in \Gamma_c^\delta(x_i)$, $\psi_z^{-1}[C_i^n, z]$ has dense interior in $\Gamma_e^\delta(z)$ and $\psi_z^{-1}[C_i^n, z] \subset \Gamma_e^{(1+\cdots+\beta^n)\delta/2}(z) \subset \Gamma_e^\delta(z)$.*

Denote $C_i = \overline{\bigcup_{n \geqq 0} C_i^n} \subset \Gamma_e^\delta(x_i)$, $C_i = \overline{\operatorname{Int} C_i}$ in $\Gamma_e^\delta(x_i)$.

LEMMA 2.4′. *If $w \in [C_i, \Gamma_c^\delta(x_i)]$, then, for some $r(w) \in J$, $t(w) \in [-q, \tau + q]$ and $[C_{r(w)t(w)}, D^0_{r(w)t(w)}]$ containing g_z^{-1}:*

$$g^{-1}\psi_w^{-1}[C_i, w] \supset \psi_{g^{-1}w}^{-1}[C_{r(w)t(w))}, g^{-1}w].$$

We now set $A_i = [C_i, D_i]$, $i = 1, \cdots, k$, $A_i = \overline{\operatorname{Int} A_i}$, $A_i = T^{-1}A_i$, $\kappa_{[-q\ \tau+q]} = \{A_i\ , t \in [-q, \tau + q], i = 1, \cdots, k\}$. Consider the system of parallelograms $\mathfrak{A} = \{A_i, i = 1, \cdots, k\}$.

Lemmas 2.4 and 2.4′ combine to give:

LEMMA 2.5. *If $y \in A_i$, then, for some $E = A_{jt_1} \in \kappa_{[-q, \tau+q]}$ containing $g(y)$, $g[y, D_i] \supset [g(y), D_{jt_1}]$, and for some $A_{rt_2} \in \kappa_{[-q, \tau+q]}$ containing $g^{-1}y$, $g^{-1}\psi_y^{-1}[C_i, y] \supset \psi_{g^{-1}y}^{-1}[C_{rt_2}, g^{-1}y]$.*

Let $V_i = \bigcup_{t=-2q}^{\tau+2q} T^{-t} A_i$. For $P \in \kappa_{[-2q, \tau+2q]}$, denote $K_p = \{i : V_i \cap P \neq \emptyset\}$. Take $i \in K_p, x \in V_i \cap P$ and A_{it} containing x for some t. Consider the projections $P_i = [C_{P_i}, D_{P_i}] \subseteq P$ of the parallelograms A_{it} on the parallelogram P. Denote $R_i = \{j \in K_P : P_i \cap P_j \neq \emptyset\}$. Let

$$Z_P = P \setminus \bigcup_{i \in K_P} \left(\bigcup_{x \in \partial C_{Pi}} D_P(x) \cup \bigcup_{y \in \partial D_{Pi}} C_P(y) \right).$$

For $y \in Z_P$, set $R(y) = \bigcup_{P_i \ni y} R_i$ and consider:

$$S_c(y) = \{j \in R(y) : [y, D_{P_j}] \ni y\} \tag{2.1}$$

$$H_c(y) = \{j \in R(y) : [y, D_{P_j}] \not\ni y\}$$

$$D(y) = \bigcap_{j \in Sc(y)} \operatorname{Int}[y, D_{P_j}] / \bigcup_{j \in Hc(y)} [y, D_{P_j}].$$

Similarly,

(2.1′)
$$S_e(y) = \{j \in R(y): [C_{P_j}, y] \ni y\}$$
$$H_e(y) = \{j \in R(y): [C_{P_j}, y] \not\ni y\}$$
$$C(y) = \bigcup_{j \in Se(y)} \mathrm{Int}[C_{P_j}, y] \setminus \bigcup_{j \in He(y)} [C_{P_j}, y].$$

Set $B(y) = [C(y), D(y)]$. Repeating the reasoning of [5, Lemmas 20, 21] we can prove:

LEMMA 2.6. *If $y, z \in Z_p$ and $B(y) \cap B(z) \neq \emptyset$, then $B(y) = B(z)$.*

LEMMA 2.7. *The family $v_p = \overline{\{B(Z), z \in Z_P\}}$ is a finite partition of P into parallelograms.*

Construct the partition v_P for all $P \in \kappa_{[-2q, \tau+2q]}$ *and denote*:

$$v = \{Q \in v_P,\ P \in \kappa_{[-2q, \tau+2q]}\},\quad Z = W^n \setminus \bigcup_{Q \in v} \bigcup_{t=0}^{m+2q} T^{-t}\, \partial Q.$$

Take $A_i \in \mathfrak{A}$, $B \in v_{A_i}$, $B_t = T^{-t}B$, $z \in B_t \cap Z$ and set:

$$\kappa(z) = \{P \ni z: P \in \kappa_{[-q, \tau+q]}\}$$
$$\kappa(B_t) = \{P \in \kappa(z),\ z \in B_t \cap z\}$$
$$v(B_t) = \{B_t Q: Q \in v_P,\ P \in \kappa(B_t)\}$$
$$J_t(z) = \bigcup_{B_t Q \ni z} B_t Q$$

where $B_t Q$ is the projection of Q on B_t.

By the construction of v_P, the family

$$\alpha(B_t) = \{J_t(z),\ z \in B_t \cap Z\}$$

is a finite partition of B_t into parallelograms. Consider the product of partitions:

$$\alpha(B) = \bigvee_{t=0}^{m} T^t \alpha(B_t).$$

For $y \in B \cap Z$, set

$$v_t(y) = \{Q_t: B_t Q_t \ni T^{-t}y,\ B_t Q_t \in v(B_t)\}$$
$$v(y) = \{Q_t \in v_t(y):\ 0 \leqq t \leqq m\}.$$

The partition $\alpha(B)$ coincides with the partition:

$$\{\bigcap_{Q_t \in v(y)} T^t(B_t Q_t) = \bigcap_{Q_t \in v(y)} B(T^t Q_t);\ y \in B \cup Z\}.$$

Consider the system of parallelograms $\mathfrak{M} = \{P \in \alpha(B),\ B \in v_{A_i},\ i = 1, \cdots, k\}$, $f_{\mathfrak{M}} = f$.

LEMMA 2.8. *The system $\mathfrak{M}$ is Markovian for the flow $\{T^t\}$.*

PROOF. Let $x \in \mathrm{Int}\, P_i \cap f^{-1}(\mathrm{Int}\, P_j) \cap Z$, P_i, $P_j \in \mathfrak{M}$, $l_{\mathfrak{M}}(x) = s (s \geqq 2q)$. Set $P_i = P(x) = [C_P(x), D_P(x)]$, $f(x) = y$, $P_j = P(f(x)) = P(y) = [C_P(y), D_P(y)]$ and assume that $P(x) \in \alpha(B(x)$, $(P(y) \in \alpha(B(y))$, $B(x) \in v_{A_k}$, $B(y) \in v_{A_r}$, A_k, $A_r \in \mathfrak{A}$.

By the definition of $P(x)$ and $P(y)$;

$$D_P(x) = \bigcap_{Q_t \in v(x)} T^t D_{Q_t}(T^{-t} x)$$

$$D_P(y) = \bigcap_{Q_t \in v(y)} T^t D_{Q_t}(T^{-t} y) =$$

$$= D_{B(y)}(y) \cap \bigcap_{\{Q_t \in v(y), t \geqq s\}} T^t_{D_{Q_t}} T^{-t} y). \tag{2.3}$$

We shall prove that in (2.3):

$$D_P(y) = \bigcap_{[Q_t \in (y) t \geqq s]} T^t D_{Q_t}(T^{-t} y). \tag{2.4}$$

To this end it is sufficient to prove that

$$D_{B(y)}(y) \supset \bigcap_{\{Q_t \in v(y) t \geqq s\}} T_t D_{Q_t}(T^{-1} y). \tag{2.5}$$

Return to the expression for $\mathrm{Int}\, B(y)$, $B(y) \in v_{A_r}$ as given by (2.1) and (2.1′). Let $n \in R(y)$ and let P_n be the projection of $A_{nt} = T^{-t} A_n$, $A_n = [C_n, D_n]$ on A_r. Then, for some $t - q \leqq t(y) \leqq t + q$, we have $T^{t(y)} y \in [\Gamma^{\delta}_e(x_n), D_n]$. By Lemma 2.5, there exists $E(y,n) \in \kappa_{[-q, \tau+q]}$ containing $T^{-m}(T^{t(y)} y)$ such that $T^{-m}[T^{t(y)} y, D_n] \supset [T^{-m+t(y)} y, D_{E(y,n)}]$.

Denoting $T^{-t(y)}[T^{t(y)} y, D_n] = [y, D_{nt(y)}]$, we get:

$$[y, D_{nt(y)}] \supset T^{m-t(y)}[T^{-m+t(y)} y, D_{E(y,n)}]. \tag{2.6}$$

Since $E(y,n) \in \kappa_{[q- \tau+q]}$, there exists $Q_{-m+t(y)} \in v_{E(y,n)}$ such that

$$[y, D_{nt(y)}] \supset T^{m-t(y)} D_{Q_{-m+t(y)}}(T^{-m+t(y)} y).$$

The inclusion (2.5) now follows from the fact that (2.6) is true for all $n \in R(y)$ and $m - t(y) > s$. This proves (2.3).

We rewrite (2.3) as:

$$D_P(y) = \bigcap_{\{Q_t \in v(x), 0 \leqq t \leqq m-s\}} T^{t+s} D_{Q_t}(T^{-t} x) =$$

$$= T^s \bigcap_{\{Q_t \in v(x), 0 \leqq t \leqq m-s\}} T^t D_{Q_t}(T^{-t} x).$$

From (2.2) and (2.7) it follows that

$$T^s D_P(x) \subset D_P(y). \tag{2.8}$$

This proves the Markov condition (1.2). The Markov condition (1.3) can be proved similarly. □

Let $\mathfrak{M}$ be a Markovian system; set $P_{\mathfrak{M}}(x) = P \cap f^{-1}Q$ for $x \in P \in \mathfrak{M}$ and $fx \in Q \in \mathfrak{M}$. Then the system $\mathfrak{B} = \{P_{\mathfrak{M}}(x), x \in M\}$ is also Markovian, and $l_{\mathfrak{B}}(x)$ is continuous on the open parallelograms of this syetm. Therefore, the closure $V(x)$ of $\bigcup_{y \in \mathrm{int} P_{\mathfrak{M}}(x)} \bigcup_{t=0}^{l_B(y)} T^t y$ is a c-parallelepiped, and $\mu = \{V(x), x \in M\}$ is a Markov partition, as required in Theorem 2.1.

REMARK. We can choose the initial system $\mathfrak{A}^0$ in such a way that for some periodic point $w_0 \in \mathrm{Int}\, P$, $P \in \mathfrak{A}_0$, we have $f_{\mathfrak{A}_0} w_0 = w_0$. Then, if the m in Lemma 2.1 is sufficiently large, it is c'ear from the construction that this property remains valid for the Markovian system $\mathfrak{B}$. It is obvious that then $w_0 \in \tilde{M}_{\mathfrak{B}}$ (see Proposition 1.2).

3. Estimation of the measure of the boundary: Hölder condition

We now assume that the flow $\{T^t\}$ is of class C^2. We shall prove, as in [14] for the case of C-diffeomorphisms, that the sets $A_i = [C_i, D_i]$ obtained by the above limit procedure are indeed parallelograms in accordance with Definition 1.2. This definition requires that the sets C_i and D_i be admissible on the leaves $\Gamma_e(x_i)$ and $\Gamma_c(x_i)$, respectively. We shall prove that the boundaries ∂C_i and ∂D_i have Lebesgue measure zero on $\Gamma_e(x_i)$ and $\Gamma_c(x_i)$. Then, obviously, the sets $C(y)$ and $D(y)$ in (2.1′), obtained from $\{C_{i_t}\}$ and $\{D_{i_t}\}$ by finite operations of intersection and complementation, will also possess this property. The same holds for the sets $\{C_P\}$, $\{D_P\}$ for all P in the Markovian system $\mathfrak{B}(\mu)$.

In this section we shall also prove that the function $l_{\mathfrak{B}(\mu)}$ satisfies a Hölder condition on every continuity component.

We first indicate a few auxiliary facts concerning the properties of C-flows of class C^2 (see [2], [4]). Let $\Delta_c^t(y)$ denote the Jacobian of the transformation taking Riemannian volume S_c on $\Gamma_c(y)$ into Riemannian volume on $\Gamma_c(T^t y)$. Since W^n is compact, there exists a constant $K_1 > 0$ such that for all $y \in W^n$ and $t \in [0, 1]$

$$K_1 \leqq \Delta_c^t(y) \leqq 1.$$

Since the flow is of class C^2, $\Delta_c^t(y)$ satisfies a Lipschitz condition on $\Gamma_c(y)$, with a constant independent of y. Therefore, for all $t \in [0,1]$, $y_1, y_2 \in \Gamma_c(y)$, and a constant $K_2 > 0$,

$$\left| \frac{\Delta_c^t(y_1)}{\Delta_c^t(y_2)} - 1 \right| \leqq K_2 d_c(y_1, y_2). \tag{3.1}$$

Now let $D \subset \Gamma_c$, $\operatorname{diam}_c D < \infty$, and let $\{t_k\}$ be an increasing sequence of positive numbers tending to ∞, $t_0 = 0$, $t_i - t_{i-1} \leqq 1$. For $y_1, y_2 \in D$, consider the quotient

$$\Delta_c^n(y_1, y_2) = \frac{\Delta_c^{t_1}(y_1)\Delta_c^{t_2-t_1}(T^{t_1}y_1), \cdots, \Delta_c^{t_n-t_{n-1}}(T^{t_{n-1}}y_1)}{\Delta_c^{t_1}(y_2)\Delta_c^{t_2-t_1}(T^{t_1}y_2), \cdots, \Delta_c^{t_n-t_{n-1}}(T^{t_{n-1}}y_2)}.$$

Since $d_c(T^{t_i}y_1, T^{t_i}y_2) < c \operatorname{diam}_c D \cdot \lambda^{t_i}$, it follows from (3.1) that

$$\left| \frac{\Delta_c^{t_{i+1}-t_i}(T^{t_i}y_1)}{\Delta_c^{t_{i+1}-t_i}(T^{t_i}y_2)} - 1 \right| \leqq K_3 \operatorname{dima}_c D \cdot \lambda^{t_i}$$

where $K_3 = cK_2$.

Therefore,

$$\Delta_c^-(\operatorname{diam}_c D) = \prod_{(j=1)}^{\infty} (1 - K_3 \cdot \lambda^{t_j} \operatorname{diam}_c D) \leqq \Delta_c^n(y_1, y_2) \leqq \tag{3.2}$$

$$\leqq \prod_{j=1}^{\infty} (1 + K_3 \cdot \lambda^{t_j} \operatorname{diam}_c D) = \Delta_c^+(\operatorname{diam}_c D).$$

Since $\{t_k\}$ increases as $k \to \infty$, the product in (3.2) is convergent.

Similarly, let $\Delta_e^t(z)$ be the Jacobian of the transformation at z taking Riemannian volume on $\Gamma_e(z)$ into Riemannian volume on $\Gamma_e(T^{-t}z)$ and $C \subset \Gamma_e$, $\operatorname{diam}_e C < \infty$ and $z_1, z_2 \in C$. Then

$$\Delta_e^-(\operatorname{diam}_e C) = \prod_{j=1}^{\infty} (1 - K_4 \cdot \lambda^{t_j} \operatorname{diam}_e C) \leqq \Delta_e^n(z_1, z_2) \leqq$$

$$\leqq \prod_{j=2}^{\infty} (1 + K_4 \cdot \lambda^{t_j} \operatorname{diam}_e C) = \Delta_e^+(\operatorname{diam}_e C)$$

where $K_4 > 0$ is a constant.

Now, for admissible sets $A, D \subset \Gamma_c^y(x)$, consider

$$\frac{S_c(T^{t_n}A)}{S_c(T^{t_n}D)} = \frac{\int_A \Delta_c^{t_1}(x) \cdot \Delta_c^{t_2-t_1}(T^{t_1}x) \cdots \Delta_c^{t_n-t_{n-1}}(T^{t_{n-1}}x) dS_c(x)}{\int_D \Delta_c^{t_1}(x)\Delta_c^{t_2-t_1}(T^{t_1}x) \cdots \Delta_c^{t_n-t_{n-1}}(T^{t_{n-1}}\ x) dS_c(x)}$$

It follows from (3.2) that for any t_n

$$\frac{\Delta_c^-(\gamma)}{\Delta_c^+(\gamma)} \cdot \frac{S_cA}{S_cD} \leqq \frac{S_c(T^{t_n}A)}{S_c(T^{t_n}D)} \leqq \frac{\Delta_c^+(\gamma)}{\Delta_c^-(\gamma)} \cdot \frac{S_cA}{S_cD}.$$

Since $\Delta_c^-(\gamma)$ and $\Delta_c^+(\gamma)$ tend to unity as $\gamma \to 0$, we have

$$\left| \frac{S_c(T^tA)}{S_c(T^tD)} - \frac{S_cA}{S_cD} \right| < \phi_c(\gamma) \frac{S_cA}{S_cD}$$

where $\phi_c(\gamma) \to 0$ as $\gamma \to 0$, $t > 0$.

Similarly, for admissible B, $C \subset \Gamma_e^\gamma(x)$,

$$\left| \frac{S_e(T^{-t}B)}{S_e(T^{-t}C)} - \frac{S_eB}{S_eC} \right| < \phi_e(\gamma) \cdot \frac{S_eB}{S_eC}$$

where $\lim_{\gamma \to 0} \phi_e(\gamma) = 0$, $t > 0$.

Consider on $\Gamma_c^\gamma(x)$ the π_e-isomorphism $\Gamma_c^\gamma(x) \to [y, \Gamma_c^\gamma(x)]$ (for the notation, see Section 1), $\pi_e(x) = y \in G_e^\gamma(x)$. Besides the measure S_c on $\Gamma_c^\gamma(x)$, we consider the measure $\tilde{S}_c$ obtained by transferring Riemannian volume on $\Gamma_c^\gamma(y)$ to $\Gamma_c(x)$ by the transformation π_e^{-1}. Since the foliation G_e is absolutely continuous, the measures S_c and $\tilde{S}_c$ are equivalent. Let $I(x)$ be the density of $\tilde{S}_c$ relative to S_c at the point x. The function $I(x)$ is continuous, and for every $\varepsilon > 0$ there exists $\delta_1 = \delta_1(\varepsilon)$, independent of x, such that for any two measurable sets $D_x, A_x \subset \Gamma_c^\gamma(x)$ $S_c(D_x) > 0$, $D_y = [y, D_x]$, $A_y = [y, A_x]$, if $d_e(x,y) < \delta_1$ then

$$\left| \frac{S_cA_x}{S_cD_x} - \frac{S_cA_y}{S_cD_y} \right| < 2\varepsilon \frac{S_cA_x}{S_cD_x}.$$

Similarly, for measurable sets B_u, $C_u \subset \Gamma_e^\gamma(u)$, $S_e(C_u) > 0$, $B_v = [B_u, v]$, $C_v = [C_u, v]$, $d_c(u,v) < \delta_2$,

$$\left| \frac{S_eB_u}{S_eC_u} - \frac{S_eB_v}{S_eC_v} \right| < 2\varepsilon \frac{S_eB_u}{S_eC_u}.$$

In addition, for any $0 < \tilde{\varepsilon} < 1$ there exist $\tilde{\delta} = \tilde{\delta}(\tilde{\varepsilon}) < \gamma$ and $\tilde{t} = \tilde{t}(\tilde{\varepsilon}) > 0$ such that for all x, y with $d_e(x,y) < \tilde{\delta}$ and u, v with $d_c(u,v) < \tilde{\delta}$, $t \in [0, \tilde{t}]$:

$$\left| \frac{S_c(T^{-t}A_y)}{S_c(T^{-t}D_y)} - \frac{S_cA_x}{S_cD_x} \right| < \tilde{\varepsilon} \frac{S_cA_x}{S_cD_x} \tag{3.4}$$

$$\left| \frac{S_eT^{-t}B_v}{S_eT^{-t}C_v} - \frac{S_eB_u}{S_eC_u} \right| < \tilde{\varepsilon} \frac{S_eB_u}{S_eC_u}.$$

We now return to the system of parallelograms $\mathfrak{A}^0 = A_i^0 = [C_i^0, D_i^0]$, $i = 1,\cdots,k\}$, with which we began the successive approximation procedure. We choose $C_i^0 \subset \Gamma_e^\alpha(x_i)$, $D_i^0 \subset \Gamma_c^\alpha(x_i)$ so that $\max_{x\in M} l_{\mathfrak{A}_0}(x) = L < \frac{1}{2}\tilde{t}$ (for the notation, see Section 2). Set:

$$\kappa^0 = \{A_{it}^0 : A_{it}^0 = T^{-t}A_i^0,\ t\in[0,2L]\quad i = 1,\cdots,k\}$$

$$\kappa_c^0 = \{D_{it}^0 : D_{it}^0(y) = T^{-t}[y, D_i^0],\ y\in C_i^0,\ t\in[0,2L],\ i = 1,\cdots,k\}$$

$$\kappa_e^0 = \{C_{it}^0 : C_{it}^0(v) = T^{-1}\psi_v^{-1}[C_i^0, v],\ v\in D_i^0,\ t\in[0,2L],\ i = 1,\cdots,k\}.$$

We choose α so that the diameters of the sets κ_c^0 and κ_e^0 are at most $\tilde{\delta}/2$. For the successive approximations D_i^n, C_i^n, we set:

$$\kappa_c^n = \{D_{it}^n : D_{it}^n(y) = T^{-t}[y, D_i^n],\ y\in C_i^0,\ t\in[0,2L],\ i = 1,\cdots,k\}$$

$$\kappa_e^n = \{C_{it}^n : C_{it}^n(v) = T^{-t}\psi_v^{-1}[C_i^n, v],\ v\in D_i^0,\ t\in[0,2L],\ i = 1,\cdots,k\}.$$

For every $n = 0,1,\cdots$, the sets of the systems κ_c^n and κ_e^n cover each complete leaf of Γ_c and Γ_e, respectively. Moreover, by construction, $T^{-m}D_i^1$ (where m is large) consists of sets of the system κ_c^0, $T^{-m}D_i^2$ of sets of $\kappa_c^1,\cdots$, $T^{-m}D_i^n$ of sets κ_c^{n-1}. Therefore $T^{-mn}D_i^n$ consists of sets of the system κ_c^0. Moreover, the boundary of the set $T^{-m(n-1)}D_i^n$ is distant at most $\gamma\cdot c\cdot\lambda^m$ from the boundary of $T^{-m(n-1)}D_i^{n-1}$. Our aim is to prove that the boundary of the limit set D_i has measure zero. To this end, we construct a decreasing sequence of sets $W_1 \supset W_2 \supset \cdots$ such that $\bigcap W_i \supset \partial D_i$ and $S_c(W_i) \leqq \rho S_c(W_{i-1})$ for some $\rho < 1$.

To do this, it will be convenient to have on each complete leaf not a cover but a partition into admissible sets such that the measure of a small neighborhood of their boundaries is uniformly small. For each parallelogram $P \subset \kappa^0$, we construct a partition ν_P^0 (see (2.1), (2.1′)) as in Lemmas 2.6 and 2.7, and set $\nu^0 = \{Q = [C_Q, D_Q] : Q \subset \nu_P^0, P \subset \kappa^0\}$. We choose the initial cover $\mathfrak{U}^0$ that for every $Q \subset \nu^0$ the sets D_Q and C_Q are connected and admissible. By the construction of the system ν_0, there exist finitely many parallelograms $Q_i \in \nu^0$ and $t_i \in [0,2L]$, $i = 1,\cdots,l$, such that $\nu^0 = \{T^{-t}Q_i : t\in[0,t_i],\ i = 1,\cdots,l\}$. The sets of the systems

$$\nu_c^0 = \{D_{Q_i}^t(y) = T^{-t}[y, D_{Q_i}],\ y\in C_{Q_i},\ t\in[0,t_i],\ i = 1,\cdots,l\}$$

and

$$\nu_e^0 = \{C_{Q_i}^t(v) = T^{-t}\psi_v^{-1}[C_{Q_i}, v],\ v\in D_{Q_i},\ t\in[0,t_i],\ i = 1,\cdots,l\}$$

form a partition of every complete leaf on Γ_c and Γ_e, respectively. The sets

$D \in \nu_c^0$ and $C \in \nu_e^0$ are admissible and have diameter at most $\tilde{\delta}/2 < \gamma/2$. For $A \subset \Gamma_c(x)$, let $U_r(A)$ denote the r-neighborhood of the boundary ∂A on the leaf $\Gamma_c(x)$. Similarly, we define $U_r(B)$ for $B \subset \Gamma_e(x)$.

LEMMA 3.1. *There exist $0 < r$, $s < \tilde{\delta}/2$ and $0 < \rho_1, \rho_2 < \frac{1}{2}$ such that for all $D \in \nu_c^0$, $C \in \nu_e^0$.*

$$\frac{S_c U_r(D)}{S_c D} < \rho_1 \tag{3.5}$$

$$\frac{S_e U_s(C)}{S_e C} < \rho_2 .$$

PROOF. Choose r_i, $0 < r_i < \tilde{\delta}/2$, so that

$$0 < \frac{S_c U_{r_i} D_{Q_i}}{S_c D_{Q_i}} < \frac{1}{4} \quad i = 1,2.$$

Then, by (3.4),

$$0 < \frac{S_c T^{-t}[y, U_{r_i} D_{Q_i}]}{S_c D^t_{Q_i}(y)} < \frac{1}{2} \tag{3.6}$$

for all $t \in [0, t_i]$, $y \in C_{Q_i}$. It is obvious that

$$\underline{r_i} = \min_{t \in [0, t_i], y \in C_{Q_i}} d_c(\partial T^{-t}[y, U_{r_i} D_{Q_i}], \partial D^t_{Q_i}(y)) > 0 .$$

Therefore, for $0 < r < \min \underline{r_i}$ we have $U_r(D^t_{Q_i}(y)) \subset T^{-t}[y, U_{r_i}(D_{Q_i})]$. It then follows from (3.6) that for all $D \in \nu_c^0$

$$0 < \frac{S_c U_r(D)}{S_c D} < \frac{1}{2}.$$

This proves the first inequality of (3.5). The proof of the second is analogous.

We now choose the number m figuring in our procedure to be so large that $3 \cdot \gamma \cdot c \cdot \lambda^m < \min(r, s)$. Then:

THEOREM 3.2. *The limit sets D_i and C_i are admissible.*

PROOF. We must show that $S_c(\partial D_i) = 0$ and $S_e(\partial C_i) = 0$. Consider the nth approximation D_i^n. The set $T^{-mn} D_i^n$ lies in the leaf $\Gamma_c(T^{-mn} x_i)$ and is the union of sets in ν_c^0. We set

$$\alpha_n = \{D : D \in \nu_c^0,\ d_c(\partial D, \partial T^{-mn} D_i^n) \leqq \gamma\}$$

and $\tilde{W}_n = \bigcup_{D \in \alpha_n} D$. The boundary $\partial(T^m \tilde{W}_n)$ is contained in the $2 \cdot \gamma \cdot c \lambda^m$-neighborhood of the boundary of $T^{-m(n-1)}D_i^{n-1}$, while the boundary $\partial(T^{-m(n-1)}D_i^n)$ is by construction in a $c\cdot\gamma\cdot\lambda^m$-neighborhood of $\partial T^{-m(n-1)}D_i^{n-1}$. Therefore $T^m\tilde{W}_n$ lies in the $3\cdot\gamma\cdot c\cdot\lambda^m$-neighborhood of $\partial T^{-m(n-1)}D_i^{n-1}$, and a fortiori (since $3\cdot\gamma\cdot c\cdot\lambda^m < r$) in its r-neighborhood. Therefore $T^m\tilde{W}_n \subset \bigcup_{D\in\alpha_{n-1}} U_r(D)$ and $T^m\tilde{W}_n \subset \tilde{W}_{n-1}$. Setting $W_n = T^m\tilde{W}_n$, we have $W_n \subset W_{n-1} \subset \cdots$. We claim that $S_c(W_n) \leqq \rho S_c(W_{n-1})$, where $\rho < 1$.

It follows from Lemma 3.1 that for $D \subset \tilde{W}_n$, $D \cap T^m\tilde{W}_{n+1} \neq \varnothing$

$$\frac{S_c(D\cap T^m\tilde{W}_{n+1})}{S_cD} \leqq \frac{S_c(U_rD)}{S_cD} < \tilde{\rho} < \frac{1}{2}.$$

We have $W_n = \bigcup_{D\in\alpha_n} T^{mn}D$ and $W_{n+1} \subset \bigcup_{D\in\alpha_n} T^{mn}U_r(D)$. Consider the quotient $S_c(T^{mn}U_rD))/S_c(T^{mn}D)$.

It follows from (3.3) that if γ is so small that $\phi_c(\gamma) < 1$, then

$$\frac{S_c(T^{mn}U_r(D))}{S_cT^{mn}D} < 2\frac{S_cU_r(D)}{S_cD} < 2\tilde{\rho} < 1.$$

Finally,

$$\frac{S_c(W_{n+1})}{S_c(W_n)} \leqq \frac{\sum_{D\in\alpha_n} S_c(T^{mn}U_r(D))}{\sum_{D\in\alpha_n} S_c(T^{mn}D)} < 2\tilde{\rho}\,\rho = \;= 1.$$

Since $\bigcap_n W_n \supset \partial D_i$, it follows that $S_c(\partial D_i) = 0$. One proves in a similar fashion that $S_e(\partial C_i) = 0$. $\square$

We shall now prove that the function $l_{\mathfrak{B}} = l$ of the Markovian system $\mathfrak{B} = \{P_i = [C_i, D_i],\ i = 1,\cdots,k\}$, $C_i \subset \Gamma_e$, $D_i \subset \Gamma_c$, satisfies a Hölder condition on the open parallelograms $\{P_i\}$. Note that the sets C_i, D_i are in general not in P_i, and $[C_i, z]$ is c-isomorphic to C_i for any $z \in D_i$. The function l is constant on the sets $[u, D_i]$, and therefore the set of its values on $\operatorname{Int} P_i$ is the same as the set of its values on any $\operatorname{Int}[C_i, z]$. We fix $\tilde{C}_i = [C_i, z]$ and consider on $\tilde{C}_i$ the metric of the leaf G_e.

THEOREM 3.1. *The function l satisfies a Hölder condition on* $\operatorname{Int}\tilde{C}_i$.

PROOF. Let $E_{ik} = \{y \in \operatorname{Int}\tilde{C}_i : f(y) = P_k\}$ We shall prove that the Hölder condition is satisfied on each E_{ik}. Let $\pi_c^i\colon \tilde{C}_i \to C_i$ be the c-isomorphism. For $x \in C_i$, we set $p(x)$ equal to the unique point in the intersection $G_c(x) \cap C_k$ nearest to x in $G_c(x)$ (recall that the sizes of the parallelograms P_k and their mutual distances are sufficiently small). Let $\pi_c^k\colon C_k \to C_k(f(y))$ be the c-isomorphism. Then $f(y) = \pi_c^k p\, \pi_c^i(y)$. The foliations G_c and Γ_c satisfy a Hölder condition; this is proved in [3]. The analogous assertion for C-diffeomorphisms was also proved in [8] and the proof carries over easily to flows. Hence each of the mappings π_c^i, p and π_c^k satisfies a Hölder condition, and the same therefore holds for f on E_{ik}. Now let D be a differentiable disk containing E_{ik} and transversal to the flow. Then $\Phi(y,t) = T^t y$ is a diffeomorphism of $D \times [-\tau,\tau]$ into W^n. Since $f(y) = T^{l(y)}y$ for $y \in E_{ik}$, it follows that $l(y)$ satisfies a Lipschitz condition in f and consequently a Hölder condition in y. Q.E.D.

REMARK. Let $x \in \mathrm{Int}\,\tilde{C}_i$, and consider $\psi_x^{-1}\tilde{C}_i \subset \Gamma_e(x)$. Let $q(z)$, $z \in \psi_x^{-1}\tilde{C}_i$, be a function defining a trajectory isomorphism ψ, i.e., $\psi(z) = T^{q(z)}z \in \tilde{C}_i$. For $u \in C_i$, we define $S(u) = G_c(u) \cup \psi_x^{-1}C_i$—the point nearest to u. Then $\psi_x = (\pi_c^i)^{-1}S^{-1}$. The mappings S^{-1} and $(\pi_c^i)^{-1}$ satisfy a Hölder condition, so that the same is true of ψ_x on $\psi_x^{-1}\tilde{C}_i \subset \Gamma_e$ in the metric of Γ_e. Therefore, as before, $q(z)$ satisfies a Hölder condition on $\psi_x^{-1}\tilde{C}_i$. In the sequel we shall sometimes consider the metric on $\tilde{C}_i$, carried over by ψ_x from $\psi_x^{-1}\tilde{C}_i$. It is clear that the function l will then satisfy a Hölder condition in this metric as well.

4. Symbolic dynamics

As before, let M be the set-theoretic union of the parallelograms $\{P_i = [C_i, D_i],\ C_i, D_i \subset P_i,\ i = 1,\cdots,k\}$ of a Markovian system $\mathfrak{B}$, considered in the natural topology, $f_{\mathfrak{B}} = f: M \to M$ defined by $f(x) = T^{l(x)}x$, $0 < l < l(x) < L < \infty$. We consider on M a pair of partitions $\xi_c = \{D_c: [y, D_i],\ y \in C_i\ i = 1,\cdots,k\}$ and $\xi_e = \{C_e: [C_i, v],\ v \in D_i,\ i = 1,\cdots,k\}$. We denote $E = E_c \cup E_e$, where

$$E_c = \bigcup_{P \in \mathfrak{B}} \bigcup_{t=-\infty}^{\infty} T^t \partial_c P,\ E_e = \bigcup_{P \in \mathfrak{B}} \bigcup_{t=-\infty}^{\infty} T^t \partial_e P,$$

$M_1 = M \cap E$, $\tilde{M} = M \setminus M_1$. M_1 and $\tilde{M}$ are dense in M and invariant under f, and moreover f is continuous on $\tilde{M}$. We consider on $\tilde{M}$ partitions $\tilde{\xi}_c = \{\tilde{D}_c: D_c \cap \tilde{M}\}$ and $\tilde{\xi}_e = \{\tilde{C}_e: C_e \cap \tilde{M}\}$. Since $\mathfrak{B}$ is a Markovian system, we have

$$f^n \tilde{D}_c(x) \subset \tilde{D}_c(f^n x)$$

$$f^{-n} \tilde{C}_e(x) \subset \tilde{C}_e(f^{-n} x).$$

For $n \geqq 0$, we set $\tilde{\Sigma}_c^n(x) = f^{-n}\tilde{D}_c(f^n x)$, $\tilde{\Sigma}_e^n(x) = f^n \tilde{C}_e(f^{-n}x)$. It is clear that $\tilde{\Sigma}_c^n \subset \tilde{\Sigma}_c^{n+1}$ and $\tilde{\Sigma}_e^n \subset \tilde{\Sigma}_e^{n+1}$. Let $\Sigma_c^n(x)$ and $\Sigma_e^n(x)$ be the closures in M of sets $\tilde{\Sigma}_c^n(x)$ and $\tilde{\Sigma}_e^n(x)$. We define the complete contractible leaf at x in $\tilde{M}$ to be $\tilde{\Sigma}_c(x) = \bigcup_{n=0}^{\infty} \tilde{\Sigma}_c^n(x)$. Similarly, we define $\tilde{\Sigma}_e(x)$ in $\tilde{M}$, and $\Sigma_c(x)$, $\Sigma_e(x)$—the complete leaves at x in M. The mapping $f \,|\, \tilde{M}$ takes leaves into leaves.

On the elements $D_c \subset \Gamma_c$ we shall consider the Riemannian metric d_c and Riemannian volume S_c induced by imbedding in Γ_c. Let $C_e = [C_i, v]$, $v \in D_i$, $x, y \in C_e$ and $x_1 = \psi_v^{-1}x$, $y_1 = \psi_v^{-1}y$. Since ψ satisfies a Hölder condition of some order α, it follows that $d_e'(x,y) \leqq A[d_e(x_1,y_1)]^\alpha$, where $A > 0$ is a constant, d_e' the metric in $G_e \supset C_e$ and d_e the metric in $\Gamma_e(v)$ with A and α the same for all $C_e \in \xi_e$. In the sequel it will be convenient to consider the metric and Riemannian volume on C_e carried over from $\psi_v^{-1}C_e$ by ψ_v. This will enable us to make use of the convenient relation (4.1) for f^{-1}, a contraction on C_e analogous to the contraction in the case of C-diffeomorphisms, without the exponent of the power on $d_e(x,y)$. In this new metric, $d_e(x,y)$ stands for $d_e(x_1,y_1)$. As before, let $q = \max_{x,y}|q_x(y)|$, where $q_x(y)$ is defined by $T^{q_x(y)}y = \psi_x^{-1}y$ for $x, y \in C_e$, $2q < l$ and $\tilde{\lambda} = \lambda^{l-2q}$. Then:

$$
(4.1)\qquad \begin{aligned} &d_c(f^n x, f^n y) \leqq c\tilde{\lambda}^n d_c(x,y) \quad \text{if } x, y \in \operatorname{Int} D_c \\ &d_e(f^{-n}x, f^{-n}y) \leqq c\tilde{\lambda}^n d_e(x,y) \quad \text{if } x, y \in \operatorname{Int} C_e. \end{aligned}
$$

Consider the partitions $\beta = \{P \in \mathfrak{B}\}$ on M and $\tilde{\beta} = \{P \cap \tilde{M},\ P \in \mathfrak{B}\}$ on $\tilde{M}$. We introduce symbolic dynamics for f (see [5], [7]).

For $P, Q \in \beta$, we define

$$
\alpha(P,Q) = \begin{cases} 1, & \text{if } f(\operatorname{Int} P) \cap \operatorname{Int} Q \neq \varnothing, \\ 0 & \text{otherwise}. \end{cases}
$$

Let $\beta^{\mathbb{Z}}$ be the set of all bilaterally infinite sequences $(P_i)_{i=-\infty}^{\infty}$ and $\Omega(\beta) \subset \beta^{\mathbb{Z}}$ the set of all such sequences for which $\alpha(P_i, P_{i+1}) = 1$. We define a metric in $\beta^{\mathbb{Z}}$ by setting $d(\mathbb{P},\mathbb{Q}) = \sum_{i \in \mathbb{Z}} 2^{-|i|}\rho(P_i,Q_i)$, where

$$
\rho(P_i,Q_i) = \begin{cases} 0, & \text{if } P_i = Q_i, \\ 1, & \text{if } P_i \neq Q_i. \end{cases}
$$

Then Ω is a closed subset of the compact metric space $\beta^{\mathbb{Z}}$. We define the shift homeomorphism $\sigma: \Omega \to \Omega$ by $(\sigma(\mathbb{P}))_i = P_{i-1}$. We shall show later that σ is a topological mixing.

LEMMA 4.1. *There exists* $\delta > 0$ *such that if* diam C_P, diam $D_P < \delta$ *for* $P \in \beta$, *then for any* $x, y \in \tilde{M}$ *there exists* m *such that* $f^m x$ *and* $f^m y$ *lie in different parallelograms of* β.

PROOF. For $\gamma > 0$, we have

$$
G_c^\gamma(x) \supset \{y \in W^n : d(T^t x, T^t y) \leqq \gamma \quad t \geqq 0\}
$$

$$G_e^\gamma(x) \supset \{y \in W^n : d(T^t x, T^t y) \leqq \gamma \quad t \leqq 0\}$$

where d is the metric in W^n. Let γ be such that $G_e^\gamma(x) \cap G_c^\gamma(x) \subset \bigcup_{t=-\gamma}^{\gamma} T^t x$ for all $x \in W^n$. Then for any $y \notin \bigcup_{t=-\gamma}^{\gamma} T^t x$ there exists t such that $d(T^t x, T^t y) > \gamma$, since otherwise $y \in G_c^\gamma(x) \cap G_e^\gamma(x) \subset \bigcup_{t=-\gamma}^{\gamma} T^t x$. Now let δ and γ be so small that if $\operatorname{diam} C_P$, $\operatorname{diam} D_P < \delta$ and $x, y \in P$, then $d(T^t x, T^t y) \leqq \gamma$ for all $t \in [-L, L]$ $(l(z) < L)$ and $P \cap \bigcup_{t=-\gamma}^{\gamma} T^t u = u$ for all $u \in P$. Now, if for $u, v \in \tilde{P}$ we have $f^m u$ and $f^m v$ in the same parallelogram for all m, then for all t we have $d(T^t u, T^t v) < \gamma$, so that $v \in \bigcup_{t=-\gamma}^{\gamma} T^t u$, contradicting the choice of δ and γ. This proves the lemma.

Henceforth we shall consider parallelograms of sizes not exceeding δ.

LEMMA 4.2. *For $\mathbb{P} = \{P_i\}_{i=-\infty}^{\infty} \in \Omega$, the intersection $\bigcup_{-\infty}^{\infty} \overline{f^{-i} \operatorname{Int} P_i}$ contains a single point.*

PROOF. We first prove that the intersection $\bigcap_{-\infty}^{\infty} \overline{f^{-i} \operatorname{Int} P_i}$ is not empty. Denote $A_n = \bigcap_{-n}^{n} \overline{f^{-i} \operatorname{Int} P_i} = \bigcap_{-n}^{0} \cap \bigcap_{0}^{n} = A_n'' \cap A_n'$. Since the system of parallelograms is Markovian, it follows that for any $\tilde{C}_e(x) \subset P_i$ the intersection ${}^{-1}\tilde{C}_e(x) \cap P_{i-1}$ is not empty and $f^{-1} \tilde{C}_e(x) \subset \tilde{C}_e(f^{-1}x)$. Therefore, for each n the set $A_n'' \subset P_0$ is not empty and for any $x \in A_n''$ we have $C_e(x) \subset A_n''$. Similarly, A_n' is not empty for any n and for any $x \in A_n'$ we have $D_c(x) \subset A_n' \subset P_0$.

In the parallelogram P_0, each element D_c intersects each C_e, and so $A_n'' \cap A_n' = A_n \neq \varnothing$. We thus obtain a decreasing sequence $A_0 \supset A_1 \supset \cdots$ of compact sets, and so $\bigcap_{n>0} A_n = \bigcap_{-\infty}^{\infty} \overline{f^{-i} \operatorname{Int} P_i}$ is not empty.

Note that each A_n is a parallelogram. Now let $x, y \in \bigcap_{-\infty}^{\infty} \overline{f^{-i} \operatorname{Int} P_i}$ and $u_n, v_n \in \tilde{M}$ such that $u_n, v_n \in A_n$ and $u_n \to x$, $v_n \to y$. Then for all $-n \leqq m \leqq n$ and all $k \geqq n$ we have $f^m u_n$ and $f^m v_n$ in the same parallelogram $P_m \in \beta$. By Lemma 4.1, it follows that $d(T^t u_k, T^t v_k) < \gamma$ for $-Ln \leqq t \leqq Ln$ and $k \geqq n$; hence, since T is continuous, $d(T^t, T^t y) \leqq \gamma$ for $-Ln \leqq t \leqq Ln$. This is true for all n; hence for all t we have $d(T^t x, T^t y) \leqq \gamma$ and so $x = y$. This proves the lemma.

Define a mapping $\pi : \Omega \to M$ by $\pi(\mathbb{P}) = \bigcap_{i=-\infty}^{\infty} \overline{f^{-i} \operatorname{Int} P_i}$.

THEOREM 4.1. *The mapping π is continuous and "onto" and $f \circ \pi = \pi \circ \sigma$*

PROOF. Let $\mathbb{P}_k$ be a convergent sequence in Ω. This means that for every large n there exists N such that for all $k \geqq N$, $(\mathbb{P}_k)_i = (\mathbb{P}_N)_i$ for all i, $-n \leqq i \leqq n$. Then $\pi(\mathbb{P}_k) \in \bigcap_{-n}^{n} \overline{f^{-i} \operatorname{Int}(\mathbb{P}_N)_i} = A_n$ for $k \geqq N$. It is clear from (4.1) that the sizes of the parallelograms A_n tend to zero as $n \to \infty$. Therefore the sequence $\pi(\mathbb{P}_k)$ is convergent in M. This proves continuity.

Now let $x \in \tilde{M}$ such that $f^i(x) \in P_i(x)$. Then $f^i(x) \in \operatorname{Int} P_i(x) \cap f^{-1} \operatorname{Int} P_{i+1}(x))$ and consequently $\mathbb{P}(x) = \{P_i(x)\} \in \Omega$. Since $x \in \cap f^{-i} \operatorname{Int} P_i(x)$, it follows from Lemma 4.2 that $\pi(\mathbb{P}(x)) = x$. Thus $\tilde{M} \subset \pi(\Omega)$ and $\tilde{M}$ is dense in M. Since $\pi(\Omega)$ is compact, this implies that $\pi(\Omega) = M$. The relation $f \circ \pi = \pi \circ \sigma$ follows from the definitions of π and σ. □

REMARK 1. It is evident from the proof that π^{-1} is defined on $\tilde{M}$.

REMARK 2. Consider on $\tilde{M}$ the partition $\tilde{\beta}^-: \tilde{\beta}^- = \bigvee_{n=0}^{\infty} f^{-n} \tilde{\beta}$. The proof of Lemma 4.2 shows that $\bigvee_{n=0}^{\infty} f^n \tilde{\beta}^- = \varepsilon$ (the partition into points on $\tilde{M}$) and $f\tilde{\beta}^- > \tilde{\beta}^-$, $\tilde{\beta}^- = \tilde{\xi}_c$. (See [12]).

We shall now prove that the transformation f is transitive. Let $w_0 \in \tilde{M}$ be a periodic point of the flow $T^t: f(w_0) = w_0$, $T^{l_0}_{w_0} = w_0$, $l_0 > 0$ (see the remark at the end of Section 2). By the Markov property, $f^{-1} D_c(w_0) \supset \operatorname{Int} D_c(w_0)$. Consider the leaves

$$\Sigma_c(w_0) = \overline{\bigcup_{i=0}^{\infty} f^{-i} \tilde{D}_c(w_0)}$$

$$\Sigma_e(w_0) = \overline{\bigcup_{i=0}^{\infty} f^{i} \tilde{C}_e(w_0)}.$$

LEMMA 4.3. *The leaves $\Sigma_c(w_0)$ and $\Sigma_e(w_0)$ are dense in M.*

PROOF. Note first that if $y \in \tilde{M}$ lies on a trajectory $T^t y_0$, $y_0 \in \tilde{D}_c(w_0)$, then obviously $y \in \Sigma_c(w_0)$.

Let $O_c(w_0)$ be the connected component of $D_c(w_0)$ containing w_0: $O_c(w_0) = \overline{\operatorname{Int} O_c(w_0)}$ in $\Gamma_c(w_0)$. By the Markov property, $f \operatorname{Int} O_c(w_0) \subset O_c(w_0)$, or $T^{-l_0} O_c(w_0) \supset O_c(w_0)$ and by the expansion property

$$\lim_{k \to \infty} (T^{-kl_0} O_c(w_0)) = \Gamma_c(w_0),$$

where $k > 0$ is an integer. $\tilde{\Gamma}_c = \lim(T^{-kl_0} \tilde{O}_c(w_0))$ is dense in $\Gamma_c(w_0)$.

Let $x \in \operatorname{Int} P$ and let $O(x)$ be a neighborhood of x in P. We denote $O^\varepsilon(x) = \bigcup_{\tau=0}^{\varepsilon} T^t O(x)$ for small ε. Since $\Gamma_c(w_0)$ is dense in W^n, there exists $z \in \tilde{\Gamma}_e(w_0)$ such that $z \in O^\varepsilon(x)$. Then $y = T^{-\tau} z \in O(x)$ for some $0 \leqq \tau \leqq \varepsilon$. The neighborhood $\tilde{O}_c(w_0)$ contains a point y_0 such that $T^{-kl_0} y_0 = z$ for some $k > 0$. The points y_0 and y lie on the same trajectory, therefore $y \in \tilde{M}$ and $y \in \tilde{\Sigma}_c(w_0)$. The proof that $\Sigma_e(w_0)$ is dense is analogous. □

REMARK. It is clear from the proof that $\Sigma_c'(w_0) = \bigcup_{n=0}^{\infty} f^{-n} \tilde{O}_c(w_0) \subset \Sigma_c(w_0)$ is dense in M.

Let $y \in \tilde{C}_e(w_0)$ such that $D_c(y)$ and $D_c(w_0)$ are e-isomorphic. Since f is continuous on $\tilde{M}$, it follows from the Markov property that $\Sigma_c^m(w_0)$ and $\overline{f^{-m}\tilde{D}_c(y)}$ are e-isomorphic. Moreover,

$$\max_{z \in \Sigma_c^m(w_0)} d_e(z, \pi_e(z)) = d_e(\Sigma_c^m(w_0), \overline{f^{-m}\tilde{D}_c(y)}) \leqq$$

$$\leqq c d_e(D_c(w_0), D_c(y)) \,.\, \tilde{\lambda}^m .$$

DEFINITION. f is said to be a topological mixing on M if, for any two nonempty open sets U_1, U_2, there exists $k_0 > 0$ such that $f^k U_1 \cap U_2 \neq \emptyset$ for all $k \geqq k_0$.

THEOREM 4.2. *f is a topological mixing on M.*

PROOF. Since $\tilde{\Sigma}_c(w_0)$ and $\tilde{\Sigma}_e(w_0)$ are dense in M, there exist $m, n > 0$, $z_1 \in U_1$, $z_2 \in U_2$ such that $z_1 \in \tilde{\Sigma}_c^m(w_0)$, $z_2 \in \tilde{\Sigma}_e^n(w_0)$. Let r be the distance from z_1 to $\partial(C_e(z_1) \cap U_1)$ in the metric d_e. Let $\tilde{O}_c(z_2)$ be the component of $\tilde{D}_c(z_2) \cap U_2$ containing z_2. Then $y = f^{-n}z_2 \in \tilde{O}_e(w_0)$ and the intersection $V_c(y) = \tilde{O}_c(y) \cap f^{-n}\tilde{O}_c(z_2)$ is open in $\tilde{O}_c(y)$ and contains y ($\tilde{O}_c(y), \tilde{O}_e(w_0)$ are components in $\tilde{D}_c(y)$, $\tilde{C}_e(w_0)$, respectively).

Since the leaves of D_c are expanded by f^{-1}, there exists $q > 0$ such that that $f^{-q}V_c(y) \supset \tilde{O}_c(f^{-q}y)$. $\tilde{O}_c(f^{-q}y)$ is e-isomorphic to $\tilde{O}_c(w_0)$. For all N, Σ_c^N and $\overline{f^{-N}\tilde{O}_c(f^{-a}y)}$ are e-isomorphic, and if $N > N_0$, where $c\tilde{\lambda}^{N_0} d_e(O_c(y), O_e(w_0)) < r$, then

$$d_e(\Sigma_c^N, \overline{f^{-N}\tilde{O}_c(f^{-q}y)}) < r .$$

Therefore, for all $N > \max(m, N_0)$ there exists $u^N \in f^{-N}\tilde{O}_c(f^{-q}y)$ such that $d_e(u^N, z_1) < r$, and so $u^N \in U_1$. Since

$$f^{-N}\tilde{O}_c(f^{-q}y) \subset f^{-(N+q)}V_c(y) \subset f^{-(q+N+n)}\tilde{O}_c(z_2),$$

it follows that $f^{N+q+n}u^N \in \tilde{O}_c(z_2) \subset U_2$. Now, if $k_0 = N + q + n$, then for all $k \geqq k_0$,

$$f^k U_1 \cap U_2 \neq \emptyset .$$

This completes the proof. □

THEOREM 4.3. *The shift homeomorphism σ on Ω is a topological mixing.*

PROOF. (See Bowen [5]). Let U, V be nonempty open sets in Ω. For some n there exist $(2n+1)$-tuples $(F_{-n}, \cdots, F_n)$ and $(G_{-n}, \cdots, G_n)$, F_i, $G_i \in \beta$, such that

$$U \supset U_1 = \{\mathbb{P} \in \Omega \colon P_i = F_i \ \left|i\right| \leqq n\} \neq \emptyset$$

$$V \supset V_1 = \{\mathbb{P} \in \Omega \colon P_i = G_i \ \left|i\right| \leqq n\} \neq \emptyset .$$

Let $U_2 = \bigcap_{k=-n}^{n} f^{-k} \mathrm{Int} F_k = V_2 = \bigcap_{k=-n}^{n} f^{-k} \mathrm{Int}\, G_k$. U_2 abd V_2 are open in M. As in Lemma 4.2, U_2 and V_2 are not empty. $\pi^{-1}(U_2) \subset U_1$ and $\pi^{-1}(V_2) \subset V_1$. By Theorem 4.1,

$$\sigma^k U \bigcap V \supset \sigma^k U_1 \bigcap V_1 \supset \sigma^k(\pi^{-1} U_2) \bigcap \pi^{-1} V_2 \supset \pi^{-1}(f^k U_2 \bigcap V_2).$$

By Theorem 4.2, there exists $k_0 > 0$ such that $f^k U_2 \bigcap V_2^k \neq \emptyset$ for $k \geqq k_0$, and then also $\sigma^k U \cap V \neq \emptyset$. □

We now define a function F on Ω by $F(\omega) = l(\pi(\omega))$. It is clear that F satisfies a Hölder condition on Ω. Set

$$\tilde{W} = \{(\omega, t) \colon \omega \in \Omega, \ 0 \leqq t < F(\omega), \ (\omega, F(\omega)) = (\sigma\omega, 0)\}$$

and consider the natural topology on $\tilde{W}$ induced by the direct product $\Omega \times t$. We define a flow in $\tilde{W}$ by

$$S^t(\omega, u) = \begin{cases} (\omega, u + t), \ t < F(\omega) - u \\ (\sigma\omega, u + t - F(\omega)) \quad t \geqq F(\omega) - u \end{cases}$$

for $t < \inf \omega_{\in \Omega} F(\omega)$.

For other values of t, S^t is uniquely determined by the condition that $\{S^t\}$ be a one-parameter group of transformations. The mapping $\phi \colon \tilde{W} \to W$ defined by $\phi(\omega, t) = T^t \pi(\omega)$ is continuous. Moreover, as shown above, the set H of all points $w \in W$ at which ϕ^{-1} is not well defined has Lebesgue measure 0. If μ is an S^t-invariant measure on $\tilde{W}$ such that $\mu(\phi^{-1} H) = 0$, we can carry it over to W by means of ϕ, $\phi\mu = \nu$, thus getting an isomorphism of the flows T^t in (W, ν) and S^t in $(\tilde{W}, \mu)$. This was the method used by Sinai in [15] to construct Gibbs measures for C-flows, on the assumption that Markov partitions exist.

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